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INVESTIGATION OF THE FORMULATION OF THE BOUNDARY-VALUE PROBLEM OF THE LOCAL THEORY OF ELASTOPLASTIC PROCESSES†

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For a complex stressed state, a three-term defining relation [1, 2] is used which implies that the five-dimensional stress, stress rate and strain rate vectors are coplanar. On the hypothesis of local definiteness [3], the two coefficients occurring in the three-term relation are taken as functions of three functionals of the process—the stress intensity, the length of the arc of the deformation path and the angle of approach. For bounded derivatives of these two functions with respect to each argument, conditions securing a correct formulation of the static boundary-value problem in terms of rates of each instant of the elastoplastic process are determined.

A formulation is given of the quasistatic global boundary-value problem for the whole process. It is proved that the operator of the global problem, an operator of the variational calculus [4], is positive definite, strictly monotonic in the main and possesses the $(S)_{+}$-property [5]. Using the theorem of Leray and Lions [4], it is shown that a generalized solution exists. It is proved that the global solution is unique and continuously dependent on the external loads. For the step method, using discretization of the process with respect to the load parameter, and iterational methods (of the type of SN-EVM method [2]), convergence of the approximate solutions to the exact solution of the global problem is proved.

1. FOR THE defining relation for elastoplastic materials in the case of complex stressed state, a three-term relation between the deviators of the stresses S_{ij} and strains e_{ij} was proposed in [1, 2]:

$$S_{ij}^{\cdot}={}^2/_3Ne_{ij}^{\cdot}+MS_{ij}s^{\cdot}/\sigma,\quad \sigma=({}^3/_2S_{ij}S_{ij})^{1/2},\quad s^{\cdot}=({}^2/_3e_{ij}^{\cdot}e_{ij}^{\cdot})^{1/2} \tag{1.1}$$

Here and below the summation is over repeated indices i, j; the dot denotes the right-hand derivative with respect to the parameter $t\in[0, T]$, which increases monotonically with time; N and M are functions of the curvature of the deformation path over the length of its arc s, in the elastic region with $e_u\equiv(2/3e_{ij}e_{ij})^{1/2}<e_s$ we have $s\equiv e_u$, $M\equiv 0$, $N\equiv 3G$ (G is the shear modulus and e_s is the yield point).

Relation (1.1) (in vector form) was derived in [1] subject to the conditions of local definiteness [3] and the existence of an instantaneous limiting surface which is regular at the load point. Relation (1.1) was proved for an arbitrary complex load on the basis of an analysis of the experimental data, with N a function of two arguments: σ and the angle of approach $\theta=\arccos(e_{ij}^{\cdot}S_{ij}/\sigma s^{\cdot})$. Specific modifications of (1.1) were considered in [7–10].

We shall assume that the coefficients N and M are functions of the three functionals of the process σ, s and $z=\cos\theta$. Comparison with experimental data supports this hypothesis [6].‡ Defining the deviators

$$b_{ij}^{*}={}^2/_3[M(\sigma, s, z)n_{ij}+N(\sigma, s, z)p_{ij}],\quad p_{ij}=e_{ij}^{\cdot}/s^{\cdot},\quad n_{ij}={}^3/_2S_{ij}/\sigma,$$

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‡See also: YERMAKOV S. V., Analysis of the relations and boundary-value problems of the theory of elastoplastic processes of moderate and low curvature. Author's abstract, MGU, Moscow, 1984.

we rewrite (1.1) in the form of a differential equation with respect to $S_{ij}^{\cdot}$.

$$S_{ij}^{\cdot}=b_{ij}(S_{km},\ s(t),\ e_{km}^{\cdot}(t)),\ S_{ij}(0)=0;\ b_{ij}=b_{ij}^{*}(S_{km},\ s,\ p_{km})s^{\cdot} \tag{1.2}$$

We impose on the functions N, M and their derivatives the conditions

$$\begin{gathered}-3G_p\leqslant P(\Phi(s),\ s,\ -1)\leqslant P(\Phi(s),\ s,\ z)\leqslant P(\Phi(s),\ s,\ 1)=\\ =\Phi'(s)\geqslant\gamma_1>0\end{gathered} \tag{1.3}$$

$$0<N\leqslant N_0,\quad 0\leqslant|M|\leqslant M_0\sigma_0,\quad M_0=N_0-\gamma_1$$

$$|\partial E/\partial\sigma|\leqslant E_1,\quad |\partial E/\partial s|\leqslant E_2,\quad |\partial E/\partial z|\leqslant E_3\sigma$$

γ_1, M_i, N_i = const>0, (i = 0, 1, 2, 3), $\sigma_0\equiv\sigma/\Phi(s)$, $P(\sigma, s, z)\equiv M+Nz=d\sigma/ds$, where $G_p\leqslant G$ is the unloading modulus, Φ is a function of hardening during simple loading and E denotes M or N. From (1.3) we obtain

$$\sigma_0\leqslant 1;\ |b_{ij}^{*}|\equiv({}^3/_2 b_{ij}^{*}b_{ij}^{*})^{1/2}\leqslant b_0\equiv M_0+N_0,\ |b_{ij}|\leqslant b_0 s^{\cdot} \tag{1.4}$$

$$|\Delta_1 b_{ij}|\leqslant K_1|\Delta S_{ij}|+K_2|\Delta s|,\quad |\Delta_2 b_{ij}|\leqslant b_2|\Delta e_{ij}^{\cdot}|$$

$$K_1=M_1+N_1+(M_0+M_3+N_3)/(Ge_s),\ K_2=M_2+N_2,\ b_2=b_0+2(M_3+N_3)$$

$$\Delta_1 b_{ij}=b_{ij}(S_{km}^{(1)},s^{(1)},e_{km}^{\cdot(1)})-b_{ij}(S_{km}^{(1)},s^{(2)},e_{km}^{\cdot(1)})$$

$$\Delta_2 b_{ij}=b_{ij}(S_{km}^{(2)},s^{(2)},e_{km}^{\cdot(1)})-b_{ij}(S_{km}^{(2)},S^{(2)},e_{km}^{\cdot(2)})$$

$$\Delta s=s^{(1)}-s^{(2)},\quad \Delta S_{ij}=S_{ij}^{(1)}-S_{ij}^{(2)},\quad \Delta e_{ij}^{\cdot}=e_{ij}^{\cdot(1)}-e_{ij}^{\cdot(2)},$$

$$|\Delta e_{ij}^{\cdot}|=({}^2/_3\Delta e_{ij}^{\cdot}\Delta e_{ij}^{\cdot})^{1/2}$$

From (1.4) using Gronwal's lemma ([5], p. 191) it follows that

$$|\Delta_1 b_{ij}(t)|\leqslant b(t)s^{\cdot(1)}(t),\quad b(t)=\min\left\{\rho(s^{(1)}(t))\int_0^t|\Delta e_{ij}(\tau)|\,d\tau,\,b_1\right\} \tag{1.5}$$

$$\rho(s)=K_1\exp(K_1 s)(K_2 s+b_2)+K_2,\ b_1=2M_0+N_0$$

Let us determine the conditions of (1) positive definiteness; (2) monotonicity; and (3) L-continuity of the connection $S_{ij}^{\cdot}\sim e_{ij}^{\cdot}$ (1.2) for all $s>e_s$, $\sigma_0\leqslant 1$.

(1) If $F(\sigma, s, z)\equiv Mz+N\geqslant\gamma_1$ for all $z\in[-1, 1]$, then

$$S_{ij}^{\cdot}e_{ij}^{\cdot}\geqslant\gamma_1 s^{\cdot 2} \tag{1.6}$$

(2) Using Sylvester's criterion, we find that if, for $\Delta\theta=\theta^{(1)}-\theta^{(2)}\neq 0$, the condition $F(\sigma, s, z)$ is satisfied and

$$4F^{(1)}F^{(2)}-[M^{(1)}z^{(2)}+M^{(2)}z^{(1)}+(N^{(1)}+N^{(2)})\cos\Delta\theta]^2>0 \tag{1.7}$$

$$M^{(k)}=M(\sigma,s,z^{(k)}),\quad N^{(k)}=N(\sigma,s,z^{(k)}),$$

$$z^{(k)}=\cos\theta^{(k)}=e_{ij}^{\cdot(k)}S_{ij}/(\sigma s^{\cdot(k)}),\qquad k=1,2$$

then $I\equiv[b_{ij}(S_{km}, s, e_{km}^{\cdot(1)})-b_{ij}(S_{km}, s, e_{km}^{\cdot(2)})]\Delta e_{ij}^{\cdot}>0$ when $\Delta e_{ij}^{\cdot}\neq 0$. Special cases of condition (1.7) have been considered earlier in [8–11]. Using the differentiability of N and M with respect to z, as in [9], we find that if, for $z\in[-1, 1]$

$$N_z'\equiv\partial N/\partial z\leqslant 0,\ M_z'\leqslant 0,\ L(\sigma,\ s,\ z)\equiv M-M_z'z+N_z'z\leqslant 0,\ z\in[-1,\ 1] \tag{1.8}$$

$$N-N_z'z+M_z'+L\geqslant\gamma_2>0,\ z\in[-1,\ 0] \tag{1.9}$$

and for $z\in\,]0, 1]$ either inequalities (1.9) and $\Psi(\sigma, s, z)\equiv L-2N_z'z\leqslant 0$ or inequalities $\Psi(\sigma, s, z)>0$, $N+M_z'+L^2/(4N_z'z)\geqslant\gamma_2$ are satisfied simultaneously, then

$$I\geqslant\gamma_2\,{}^2/_3\Delta e_{ij}^{\cdot}\Delta e_{ij}^{\cdot},\ \gamma_2=\text{const}>0 \tag{1.10}$$

(3) From the L-continuity of b_{ij} with respect to S_{km} for functions $e_{km}^{\cdot}(t)$ which are piecewise-continuous with respect to t, it follows that a unique solution of (1.2) exists. We introduce the set of tensor-functions $S_{ij}^{\cdot}$ (or $e_{ij}^{\cdot}$):

$$Z(\chi)=\{S_{ij}^{\cdot}:\ |S_{ij}^{\cdot}(t)|\leqslant\chi<\infty,\ \forall t\in[0,T]\},\ \chi=\text{const}$$

Assertion 1.1. If inequalities (1.3), (1.6) and (1.10) are satisfied, then the operator $D\in(e_{ij}^{\cdot}\in Z(\chi)\to S_{ij}^{\cdot}\to Z(b_0\chi))$ defined by (1.2) is L-continuous and has an inverse L-continuous operator

$$D^{-1}\in(S_{ij}^{\cdot}\in Z(b_0\chi)\to e_{ij}^{\cdot}\in Z(b_0\chi/\gamma_1))$$

in the sense of the norms

$$\Big(\int_0^T |e_{ij}^{\cdot}(t)|^2\,dt\Big)^{1/2},\quad \Big(\int_0^T |S_{ij}^{\cdot}(t)|^2\,dt\Big)^{1/2}.$$

Proof. The L-continuity of the operator D with constant $T\chi\rho(T\chi)+b_2$ follows from (1.4) and (1.5). According to (1.6) and (1.10), Eq. (1.2) defines an implicit function $e_{ij}^{\cdot}=\psi_{ij}(S_{km}^{\cdot},\ S_{km},\ s)$ which is L-continuous with respect to $S_{km}^{\cdot}$ with constant $V_1=\gamma_2^{-1}$. If $S_{ij}^{\cdot}\in Z(b_0\chi)$, then it follows from (1.6) that $e_{ij}^{\cdot}\in Z(b_0\chi/\gamma_1)$, and according to (1.4), (1.10) the function $|\psi_{ij}|$ is L-continuous with respect to S_{km} and s with constants $V_h=K_{i-1}\chi/\chi_2$ $(i=2,\ 3)$. Then the equation $s^{\cdot}=|\psi_{ij}(S_{km}^{\cdot}(t),\ S_{km}(t),s)|$ with initial condition $s(0)=0$ has a unique solution, that is, the operator D^{-1}, the L-continuity of which with constant $\exp(V_3T)(V_1+TV_2)$ follows from Gronwal's lemma, is defined.

By way of illustration, we consider expressions for N and M in the form [9]

$$M=[\Phi'-(\alpha+\alpha_1 z)\Phi/\lambda]\sigma_0,\ N=(r+r_1 z\sigma_0)\Phi/\lambda;\ r_1=1-r,\ \alpha_1=1-\alpha; \tag{1.11}$$

where λ, α, r are functions of s, found from experiments on plane paths of deformation where, in the elastic range of r, α, $\Phi/(3G\lambda)=1$. When $s>e_s$ for some steels and brasses (see the reference in the footnote),

$$0{,}3\leqslant\alpha\leqslant1;\quad 1\leqslant r\leqslant1+\alpha;\quad G\leqslant\Phi/\lambda\leqslant3G \tag{1.12}$$

If the derivatives λ', α', r', Φ' and Φ'' are bounded and the conditions

$$\gamma_1\leqslant\Phi/\lambda,\quad \Phi'\lambda/\Phi\leqslant\alpha\leqslant1+r-\Phi'\lambda/\Phi,\quad 1\leqslant r\leqslant1+\alpha \tag{1.13}$$

are satisfied, all the restrictions on N, M and the properties 1–3 of relation (1.2) are satisfied, with $\gamma_2=\gamma_1/2$ [9].

2. The equilibrium equations, Cauchy relations and the coupling equations (1.2) (for given values of S_{km} amd s) together with the linear relation between the spherical tensors $\sigma_{ii}=3K\varepsilon_{ii}$ and the boundary conditions define the static boundary-value problem for any $t\in[0,T]$ relative to the velocity vector $u^{\cdot}\in C^2(\Omega)\cap C^1(\bar\Omega)$ (Lagrange's equation):

$$\int_\Omega \sigma_{ij}^{\cdot}(\mathbf{u}^{\cdot})\zeta_{ij}^{\cdot}\,d\Omega=\int_\Omega F_i^{\cdot}\zeta_i^{\cdot}\,d\Omega+\int_{\Gamma_\sigma}P_i^{\cdot}\zeta_i^{\cdot}\,d\Gamma,\quad \forall\zeta^{\cdot}\in C^1(\Omega) \tag{2.1}$$

$$\sigma_{ij}^{\cdot}(\dot{\mathbf{u}})=S_{ij}^{\cdot}(\dot{\mathbf{u}})+\delta_{ij}3K\varepsilon^{\cdot},\quad \varepsilon_{ij}^{\cdot}=e_{ij}^{\cdot}+\delta_{ij}\varepsilon^{\cdot}=(\dot u_{i,j}+\dot u_{j,i})/2,$$
$$\zeta_{ij}^{\cdot}=(\zeta_{i,j}+\zeta_{j,i})/2$$

$S_{ij}(u^{\cdot})$ is defined by relations (1.2), Γ_u and Γ_σ are parts of the surface bounded by the finite region Ω occupied by the body, F_i and P_i are the components of the volume and surface forces, $\zeta^{\cdot}$ is the virtual velocity and $\zeta^{\cdot}=0$ on Γ_u.

We consider functions $u^{\cdot}\in H$, $H(\Omega)$ is the Hilbert space [12]. The left-hand side of (2.1) defines a linear functional on $\zeta^{\cdot}\in H$. From (1.4) it follows that this functional is continuous and the operator

$$\mathbf{A}\in(H\to H^*):\ \langle\mathbf{A}u^{\cdot},\zeta^{\cdot}\rangle=\int_\Omega\sigma_{ij}^{\cdot}(\mathbf{u}^{\cdot})\zeta_{ij}^{\cdot}\,d\Omega,\quad \|\mathbf{A}\|\leqslant C_1<\infty$$

is bounded. H^* is the conjugate space. From (1.4), (1.6) and (1.10) it follows that the operator $\mathbf{A}$ is positive definite, L-continuous and strictly monotonic [9] with constants C_2, C_3 and C_4 which do not depend on S_{km} or s. Suppose that the external loads satisfy the conditions

$$F_i^{\cdot}(t)\in L_{q_1}(\Omega),\ q_1>{}^6/_5;\ P_i^{\cdot}(t)\in L_{q_2}(\Gamma_\sigma),\ q_2>{}^4/_3 \tag{2.2}$$

Then Eq. (2.1) can be written [12] in the form of an equation in H^*:

$$\mathbf{A}\mathbf{u}^{\cdot}=f,\quad f\in H^* \tag{2.3}$$

According to the theorem of Minty and Browder [5, p. 96], the following assertion is true.

Assertion 2.1. Suppose that conditions (2.3), (1.3), (1.6) and (1.10) are satisfied. Then Eq. (2.3) has a unique solution $\mathbf{u}^{\cdot}\in H$, the inverse operator $\mathbf{A}^{-1}\in(H^*\to H)$ being L-continuous and strictly monotonic.

Note 2.1. From Theorem 3.3 of [5, p. 103] there follows the strong convergence of Galerkin's solutions to the solution of (2.3), and from Theorem 3.4 of [5, p.104], the exact solution is the strong limit of linear iterations.

3. Let us formulate the global quasistatic boundary-value problem. It is required to find a vector $\mathbf{u}(x)\in C^1(0,\ T;\ C^2(\Omega)\cap C^1(\bar\Omega))$ $(\mathbf{x}=\{t,\bar x\}\in Q=[0,\ T]\times\Omega)$ satisfying (2.1) in which $S_{ij}^{\cdot}(\mathbf{u}^{\cdot})(\mathbf{x})$ is understood as a mapping of the vectors $\mathbf{u}^{\cdot}(\mathbf{x})$ on the deviators over Q defined by the Cauchy relations and Eq. (1.2).

Integrating (2.1) with respect to t, we obtain the equation of virtual work

$$\int_Q \sigma_{ij}^{\cdot}(\mathbf{u})(\mathbf{x})\zeta_{ij}^{\cdot}(\mathbf{x})\,dQ=\int_Q F_i^{\cdot}(\mathbf{x})\zeta_i^{\cdot}(\mathbf{x})\,dQ+$$
$$+\int_0^T\int_{\Gamma_\sigma} P_i^{\cdot}(\mathbf{x})\zeta_i^{\cdot}(\mathbf{x})\,d\Gamma\,dt,\qquad \forall\zeta^{\cdot}\in C^1(Q) \tag{3.1}$$

We let $X=L_2(0,\ T;\ H)$ denote the Hilbert space [5, p. 159] (of classes) of functions $t\to\mathbf{u}(t)$: $]0,\ T[\to H(\Omega)$ which are measurable, take values from H, and are such that

$$\left(\int_0^T\|\mathbf{u}^{\cdot}(t)\|_H^2\,dt\right)^{1/2}\equiv\|\mathbf{u}\|_X<\infty,\qquad (u^1,u^2)_X=\int_0^T(\mathbf{u}^1(t),\mathbf{u}^2(t))_H\,dt$$

Similarly, we define the space Y:

$$\sigma_{ij}\in Y=L_2(0,T;(L_2(\Omega))^9),\qquad \|\sigma_{ij}\|_Y=\left(\int_Q|\sigma_{ij}(\mathbf{x})|^2\,dQ\right)^{1/2}$$

X_1 denotes the set $u\in X$: $s^{\cdot}(\mathbf{x})\in L_\infty(Q)$.

The left-hand side of (3.1) defines a linear continuous functional on $\zeta\in X$. For the non-linear operator $\mathbf{U}\in(X\to X^*)$ ($X^*=L_2(0;\ T;\ H^*)$ is the conjugate space [5, p. 159], defined by the tensor $\sigma_{ij}^{\cdot}(\mathbf{u})(\mathbf{x})$, we have

$$\langle\mathbf{U}\mathbf{u},\zeta\rangle_T=\int_0^T\langle\mathbf{A}_{s_{km}(t),\,s(t)}\mathbf{u}^{\cdot}(t),\zeta^{\cdot}(t)\rangle\,dt \tag{3.2}$$

where the dependence of $\mathbf{A}$ on S_{km} and s is explicitly indicated. Using the Hölder inequality and the properties of the operator $\mathbf{A}$, we find that $\mathbf{U}$ is bounded and positive-definite with the same constants as $\mathbf{A}$:

$$\|\mathbf{U}\mathbf{u}\|_{X^*}\leqslant C_1\|\mathbf{u}\|_X,\quad \langle\mathbf{U}\mathbf{u},\mathbf{u}\rangle_T\geqslant C_2\|\mathbf{u}\|_X^2 \tag{3.3}$$

We represent the operator $\mathbf{U}\mathbf{u}$ in the form $\mathbf{B}(\mathbf{u},\mathbf{u})$, where the operator $\mathbf{u}^1,\mathbf{u}^2\to\mathbf{B}(\mathbf{u}^1,\mathbf{u}^2)$ (as an operator from $X\times X$ into X^*) is defined by the expression

$$\langle \mathbf{B}(\mathbf{u}^1, \mathbf{u}^2), \zeta \rangle_T = \int_Q [b_{ij}(S_{km}{}^1(\mathbf{x}), s^1(\mathbf{x}), e_{km}{}^{*2}(\mathbf{x})) + \\ + \delta_{ij} 3K\varepsilon^{*2}(\mathbf{x})]\zeta_{ij}{}^*(\mathbf{x})dQ \tag{3.4}$$

$$s^1(t) = \int_0^t |e_{ij}{}^{*1}(\tau)|\,d\tau$$

$S_{ij}{}^1$ is the solution of (1.2) with $e_{km}{}^* = e_{km}{}^{*1}$, $s = s^1$.

Relation (3.4) defines two operators:

(1) for $\forall \mathbf{u} \in X$ $\mathbf{B}_1 \in (X \to X^*)$: $\mathbf{B}_1 \mathbf{u}^* = \mathbf{B}(\mathbf{u}^*, \mathbf{u})$;

(2) for $\forall \mathbf{u} \in X$ $\mathbf{B}_2 \in (X \to X^*)$: $\mathbf{B}_2 \mathbf{u}^* = \mathbf{B}(\mathbf{u}, \mathbf{u}^*)$.

Note 3.1. The operator $\mathbf{B}_2$ is defined on $\forall \mathbf{u} \in X$ and the operators $\mathbf{B}_1$ and $\mathbf{U}_1$ (for the time being) only on $\mathbf{u} \in X$ for which a solution of (1.2) exists.

From (3.2) there follows the L-continuity and strict monotonicity of the operator $\mathbf{B}_2$ with the same constants as $\mathbf{A}$ ($\Delta\mathbf{u} = \mathbf{u}^1 - \mathbf{u}^2$, $\mathbf{u}^1$, $\mathbf{u}^2 \in X$):

$$\forall \mathbf{u} \in X \|\mathbf{B}(\mathbf{u}, \mathbf{u}^1) - \mathbf{B}(\mathbf{u}, \mathbf{u}^2)\|_{X^*} \leqslant C_3 \|\Delta\mathbf{u}\|_X,$$

$$\langle \mathbf{B}(\mathbf{u}, \mathbf{u}^1) - \mathbf{B}(\mathbf{u}, \mathbf{u}^2), \Delta\mathbf{u} \rangle_T \geqslant C_4 \|\Delta\mathbf{u}\|_X{}^2 \tag{3.5}$$

Note 3.2. The operator $\mathbf{U}$, generally speaking, is non-monotonic.

Lemma 3.1. Let $\mathbf{u}_n \rightharpoonup \mathbf{u}$ in X as $n \to \infty$. Then

$$w_n \equiv \langle \mathbf{B}(\mathbf{u}_n, \mathbf{u}^*), \quad \mathbf{u}_n - \mathbf{u} \rangle_T \to 0, \quad \forall \mathbf{u}^* \in X$$

Proof. Putting $\mathbf{u}^{(n)} = \mathbf{u}_n - \mathbf{u}$, we have $s^{*(n)}$, $\varepsilon^{*(n)} \rightharpoonup 0$ in $L_2(Q)$. Then from (1.4) and (3.4)

$$w_n \leqslant \int_Q [b_0 s^{**}(\mathbf{x}) s^{*(n)}(\mathbf{x}) + 9K\varepsilon^{**}(\mathbf{x})\varepsilon^{*(n)}(\mathbf{x})]dQ \to 0$$

Lemma 3.2. Let $a \in (\Omega \to R^1)$ be finite almost everywhere in Ω and $v \in L_p(\Omega)$, $1 \leqslant p < \infty$, $C = \text{const} > 0$, $g_n \in R^1$: $g_n \to 0$ as $n \to \infty$; then $w_n \equiv \min\{|g_n a|, C\} v \to 0$ in $L_p(\Omega)$.

Proof. From the condition of the lemma, we have $w_n \in L_p(\Omega)$, $w_n{}^p \leqslant C^p |v|^p \in L_1(\Omega)$ and $w_n(x) \to 0$ for almost all $\bar{x} \in \Omega$. Then from the Lebesque theorem [5] $\|w_n\| \to 0$.

Lemma 3.3. Let $\mathbf{u}_n \rightharpoonup \mathbf{u}$ in X as $n \to 0$. Then

$$\Delta_1 S_{ij}{}^* \equiv b_{ij}(S_{km}^{(u_n)}, s^{(u_n)}, e_{km}^{**}) - b_{ij}(S_{km}^{(u)}, s^{(u)}, e_{km}^{**}) \rightharpoonup 0, \quad \forall \mathbf{u}^* \in X \text{ in } Y$$

Proof. Putting

$$\mathbf{u}^{(n)} = \mathbf{u}_n - \mathbf{u}, \quad g_n(\mathbf{x}) = \int_0^t s^{*(n)}(\tau, \bar{x})\,d\tau, \quad \varkappa_n(\mathbf{x}) = \min\{g_n(\mathbf{x})\rho(s^*(\mathbf{x})), b_1\} s^{**}(\mathbf{x})$$

from (1.5) for any $\zeta_{ij} \in Y$ we have

$$\int_Q \Delta_1 S_{ij}{}^* \zeta_{ij} dQ \leqslant \int_Q \varkappa_n(\mathbf{x}) |\zeta_{ij}(\mathbf{x})| dQ$$

From the Hölder inequality it follows that $g^n \to 0$ in $L_2(Q)$. Then, using the absolute continuity of the Lebesque integral, from any weakly convergent sequence which is a subsequence of $\{\varkappa_n\}$, a subsequence can be selected which converges strongly to zero, and then from Lemma 5.4 of [5, p. 20], $\varkappa_n \rightharpoonup 0$ in $L_2(Q)$.

Corollary 1. The operator $\mathbf{B}_1$ (and therefore also $\mathbf{U}$) is semi-continuous.

Corollary 2. If, apart from convergence $\mathbf{u}_n \rightharpoonup \mathbf{u}$, the condition $g_n(\mathbf{x}) \to 0$ is satisfied for almost all $\mathbf{x} \in Q$, then according to Lemma 3.2 $\Delta_1 S_{ij}{}^* \to 0$ in Y.

By direct verification, using the lemma, we can see that if conditions (1.3), (1.6) and (1.10) are satisfied, on the whole of X, $\mathbf{U}$ is a positive definite operator of the variational calculus, which is strictly monotonic in the main part and possesses the (S)-property (see [4, p. 192]).

Subject to the conditions on the external loads

$$F_i^{\cdot}\in L_2(0,\ T;\ L_{q_1}(\Omega)),\ q_1>{}^6/_5;\ P_i^{\cdot}\in L_2(0,\ T;\ L_{q_2}(\Gamma_\sigma)),\ q_2>{}^4/_3 \tag{3.6}$$

the right-hand side of (3.1) is a linear continuous functional on $\zeta\in X$: $\langle\mathbf{f},\ \xi\rangle_T$, $\mathbf{f}\in X^*$. Then from the theorem of Leray and Lions [4], we have the following assertion.

Assertion 3.1. (Existence Theorem.) If conditions (3.6), (1.3), (1.6) and (1.10), are satisfied, equation

$$\mathbf{Uu}=\mathbf{f} \tag{3.7}$$

has (at least) one solution $\mathbf{u}\in X$.

Assertion 3.2. If $\mathbf{f}\in C(0,\ T;\ H^*)$, then for the solution of (3.7) $\mathbf{u}^{\cdot}\in C(0,\ T;\ H)$.

Proof. Let $\mathbf{u}_n^{\cdot}(\bar{x})\equiv\mathbf{U}^{\cdot}(t_n,\ \bar{x})-\mathbf{U}^{\cdot}(t,\ \bar{x})$, $f_n\equiv\mathbf{f}(t_n)-\mathbf{f}(t)$; by the condition $f_n\to 0$ in H^* as $t_n\to t$. From (1.4) and the property of $\mathbf{A}$ we have for $n>n^*$:

$$C_4\|\mathbf{u}_n^{\cdot}\|_H^2\leqslant\int_\Omega|b_{ij}(S_{km}(t),\ s(t),\ e_{km}^{\cdot}(t))-b_{ij}(S_{km}(t_n),\ s(t_n),\ e_{km}^{\cdot}((t))|s_n^{\cdot}\,\Omega+$$

$$+\langle f_n,\ \mathbf{u}_n^{\cdot}\rangle\leqslant(\|w_n\|_{L_2(\Omega)}+\|f_n\|_{H^*})\|\mathbf{u}_n^{\cdot}\|,\quad w_n(\bar{x})\leqslant\min\{|t-t_n|^{1/2}K_3\Phi(\bar{x}),\ b_1)s^{\cdot}$$

$$K_3=K_1b_0+K_2,\ \Phi(\bar{x})=(\int_{t_{n^*}}^{t}s^{\cdot 2}(\tau,\ \bar{x})\,d\tau)^{1/2}\in L_2(\Omega)$$

From Lemma 3.2 $\|w_n\|_{L_2(\Omega)}\to 0$, and so $\|\mathbf{u}_n^{\cdot}\|_H\to 0$ as $t_n\to t$.

Lemma 3.4. If $\mathbf{u}_n\rightharpoonup\mathbf{u}$ in X, $\mathbf{Uu}\to\mathbf{f}$ in X^* as $n\to\infty$; then (1) $\mathbf{Uu}=\mathbf{f}$; (2) $\mathbf{u}_n\to\mathbf{u}$ in X.

Proof. (1) By the condition of the lemma $\langle U\mathbf{u}_n,\ \mathbf{u}_n\rangle_T\to\langle\mathbf{f},\mathbf{u}\rangle_T$, and then since $\mathbf{U}$ is an operator of the type (M) [14, pp. 192, 191, 184], $U\mathbf{u}=\mathbf{f}$. (2) According to (1) $\langle\mathbf{Uu}_n-\mathbf{Uu},\ \mathbf{u}_n-\mathbf{u}\rangle_T\to 0$, and then from the (S)-property $\mathbf{u}_n\to\mathbf{u}$ in X.

Assertion 3.3. If the solution of (3.3) is unique, then the inverse operator $\mathbf{U}^{-1}\in(X^*\to X)$ is continuous.

Proof. Let $\mathbf{f}_n\to\mathbf{f}$ in X^* and $\mathbf{u}_n=\mathbf{U}^{-1}\mathbf{f}_n$, $\mathbf{u}=\mathbf{U}^{-1}\mathbf{f}$. From $(3.3)_2$ it follows that the sequence $\{\mathbf{u}_n\}$ is bounded. Then from the reflexivity of X, Lemma 3.4 and Lemma 5.4 of [5, p. 20], we obtain $\mathbf{u}_n\to\mathbf{n}$ in X.

Assertion 3.4. (Uniqueness Theorem.) If (3.7) has a solution $\mathbf{u}\in X_1\subset X$, then $\mathbf{u}$ is the unique solution in X.

Proof. For $t\in[0,\ t_1]$, where $t_1\geqslant e_s/V$ ($V=$ vrai max $c^*(\mathbf{x})<\infty$), over the whole region Ω the strains are only elastic, and so the solution is unique. suppose that when $t\leqslant t_2$ the solution is unique, and when $t>t_2\geqslant t_1$ there exists a solution $\mathbf{v}\in X$; $\mathbf{v}\neq\mathbf{u}$. Putting $\mathbf{u}^*=\mathbf{uv}$, from (1.5) and (3.5) for $t\geqslant t_2$

$$C_4\int_{t_2}^{t}\|\mathbf{u}^{\cdot *}(\tau)\|_H^2\,d\tau\leqslant\int_{t_2}\int_\Omega\min\left\{\rho(s(\tau,\ \bar{x}))\int_{t_2}^{\tau}s^{\cdot *}(y,\ \bar{x})\,dy,\ b_1\right\}s^{\cdot}(\tau,\ \bar{x})\,d\Omega\,d\tau\leqslant$$

$$\leqslant(6G)^{-1}\rho(TV)V(t-t_2)\int_{t_2}^{t}\|\mathbf{u}^{\cdot *}(\tau)\|_H^2\,d\tau$$

which is untrue for small $t-t_2$.

4. We divide the segment $[0, T]$ into ν equal parts by points $t_n = \Delta \times n$, $\Delta = T/\nu$; $n = 0, \dots, \nu$. At the nth step ($n = 1, 2, \dots, \nu$) we shall seek a solution $u^{\cdot(n)} \in H$ of the equation

$$\mathbf{A}^{(n)}\mathbf{u}^{\cdot} = f^{(n)}, \quad \mathbf{A}^{(n)}\mathbf{u}^{\cdot} \equiv \mathbf{A}_{S_{km}^{(n-1)}, s^{(n-1)}}\mathbf{u}^{\cdot}, \quad f^{(n)} \equiv f(t_n) \in H^* \tag{4.1}$$

where $S_{km}^{(n-1)}(\bar{x})$, $s^{(n-1)}(\bar{x})$ are defined on the $(n-1)$th step, and $S^{(0)}, s^{(0)} = 0$.

According to assertion 2.1, Eq. (4.1) has a unique solution. For $t \in [t_{n-1}, t_n]$, we first calculate

$$e_{ij}^{\cdot}(t) = (u_{i,j}^{\cdot(n-1)} + u_{j,i}^{\cdot(n-1)})/2, \quad s(t) = s^{(n-1)} + (t - t_{n-1})\, s^{\cdot(n-1)}$$

and then for known $e_{ij}^{\cdot}(t)$ and $s(t)$ we find $S_{ij}(t)$ as the solution of (1.2) with the initial conditions $S_{ij}(t_{n-1}) = S_{km}^{(n-1)}$. Then, putting $S_{km}^{(n)} = S_{km}(t_n)$, $s^{(n)} = s(t_n)$, we transfer to the solution of (4.1), at the $(n+1)$th step. As a result of the step method, we obtain the solution $\mathbf{u}_\Delta^{\cdot}(t, \bar{x}) = \mathbf{u}^{\cdot(n)}(\bar{x})$ for $t \in]t_{n-1}, t_n]$, which is piecewise-constant with respect to t, where the function $s(\mathbf{u}_\Delta)(t, \bar{x})$ is piecewise-linear with respect to t, and the deviator $S_{ij}(\mathbf{u}_\Delta)(t, \bar{x})$ is piecewise-smooth and continuous with respect to t.

After defining the operator $\mathbf{U}_\Delta$ and the functional $\mathbf{f}_\Delta$ corresponding to the tensor $\sigma_{ij}^{\cdot\Delta}$ and loads $F_i^{\cdot\Delta}$, $P_i^{\cdot\Delta}$ which are constant with respect to t on each step:

$$\langle \mathbf{U}_\Delta \mathbf{u}_\Delta, \boldsymbol{\zeta} \rangle_T = \sum_{n=1}^{\nu} \int_{t_{n-1}}^{t_n} \langle \mathbf{A}^{(n)}\mathbf{u}^{\cdot(n)}, \boldsymbol{\zeta}^{\cdot}(t) \rangle\, dt, \qquad \langle \mathbf{f}_\Delta, \boldsymbol{\zeta} \rangle_T = \sum_{n=1}^{\nu} \int_{t_{n-1}}^{t_n} \langle f^{(n)}, \boldsymbol{\zeta}^{\cdot}(t) \rangle\, dt$$

using Theorem 1.8 of [5, p. 153], we obtain in X^*

$$\mathbf{U}_\Delta \mathbf{u}_\Delta = \mathbf{f}_\Delta \tag{4.2}$$

We shall assume that the external loads satisfy the condition

$$\mathbf{f}_\Delta \to \mathbf{f} \text{ in } X^* \quad \text{as} \quad \Delta \to 0 \tag{4.3}$$

Lemma 4.1. Suppose that as $\Delta \to 0$ for any Δ

$$s^{\cdot(n)}(\bar{x}) \leqslant \xi(\bar{x}) \in L_2(\Omega) \text{ for almost all } x \in \bar{\Omega} \text{ and all } n = 1, \dots, T/\Delta \tag{4.4}$$

Then $w_n \equiv \|\sigma_{ij}^{\cdot\Delta}(\mathbf{u}_\Delta) - \sigma_{ij}^{\cdot\Delta}(\mathbf{u}_\Delta)\|_Y \to 0$.

Proof. From (1.5) and Lemma 3.2 we have

$$w_\Delta^2 = \sum_{n=1}^{\nu} \int_{t_{n-1}}^{t_n} \int_\Omega |b_{ij}(S_{km}^{(n-1)}, s^{(n-1)}, e_{km}^{\cdot(n)}) - b_{ij}(S_{km}(\tau), s(\tau), e_{km}^{\cdot(n)})|^2\, d\Omega\, d\tau \leqslant$$

$$\leqslant T \int_\Omega \min\{\Delta^2 \rho^2 (T\xi(\bar{x}))\, \xi^2(\bar{x}),\, b_1^2\}\, \xi^2(\bar{x})\, d\Omega \to 0 \quad \text{as} \quad \Delta \to 0.$$

Corollary. If condition (4.4) is satisfied, $\|\mathbf{U}_\Delta \mathbf{u} - \mathbf{U}\mathbf{u}_\Delta\|_{X^*} \to 0$ as $\Delta \to 0$.

Assertion 4.1. (Theorem of Convergence of the Step Method.) Suppose that as $\Delta \to 0$ the functional of external loads and the solution of (4.2) that is piece-wise constant with respect to t, $\mathbf{u}_\Delta^{\cdot}(\mathbf{x})$ satisfies conditions (4.3) and (4.4); then each weakly convergent subsequence of the sequence $\{\mathbf{u}_\Delta\}$ converges (strongly) in X to the exact solution of equation (3.7); if in (4.4) $\xi \in L_\infty(\Omega)$, then $\mathbf{u}_\Delta \to \mathbf{u}$ in X [$\mathbf{u}$ is the only solution of (3.7)] and $\sigma^{\cdot}{}_{ij}^{\Delta}(\mathbf{u}_\Delta) - \sigma_{ij}^{\cdot}(\mathbf{u})$ in Y.

Proof. From the corollary of Lemma 4.1 it follows that $\mathbf{U}\mathbf{u}_\Delta \to \mathbf{f}$ in X^*; as in the proof of Assertion 3.4, allowing for the fact that when $\xi \in L_\infty$ the solution of (3.7) is unique in X, we obtain the required result.

Note 4.1. The condition of strong convergence of the sequence of tensors $\{\sigma_{ij}^{\cdot\Delta}(\mathbf{u}_\Delta)\}$ to the exact solution is given in corollary 2 of Lemma 3.3.

5. We consider the two iteration rules:

$$1.\ \mathbf{J}\mathbf{u}_n = \mathbf{J}\mathbf{u}_{n-1} - \tau(\mathbf{U}\mathbf{u}_{n-1} - \mathbf{f});\quad 2.\ \mathbf{B}(\mathbf{u}_{n-1}, \mathbf{u}_n) = \mathbf{U}\mathbf{u}_{n-1} - \tau(\mathbf{U}\mathbf{u}_{n-1} - \mathbf{f});\quad n \geqslant 1 \tag{5.1}$$

where $\mathbf{J}$ is an operator of the linear theory of elasticity such that $\langle \mathbf{Ju}, \mathbf{u}\rangle_T = \|\mathbf{u}\|_{X^2}$.

Lemma 5.1. Suppose that (1) iterations $\mathbf{u}_n$ given by the second rule of (5.1) belong to $X_C \subset X$; (2) there exist numbers $0<p<1$, $0<\tau^*\leqslant 1$ such that in a neighbourhood of $\mathbf{u}_0$ of radius $r = \|\mathbf{Uu}_0 - \mathbf{f}\|_{X^*}/[1-p)C_4]$

$$\|\mathbf{B}(\mathbf{u}, \mathbf{v}) - \mathbf{Uv}\|_{X^*} \leqslant p\|\mathbf{B}(\mathbf{u}, \mathbf{v}) - \mathbf{Uu}\|_{X^*}, \ \forall \mathbf{u}, \mathbf{v} \in X_c: \|\mathbf{u}-\mathbf{v}\|_X \leqslant \tau^*(1-p)r \tag{5.2}$$

Then for any $\tau \in]0, \tau^*[$, $\mathbf{u}_n$ converges strongly to the solution $\mathbf{u}$ of (3.7):

$$\|\mathbf{Uu}_n - \mathbf{f}\|_{X^*} \leqslant k^n \|\mathbf{Uu}_0 - \mathbf{f}\|, \quad \|\mathbf{u}_n - \mathbf{u}\|_X \leqslant k^n r, \quad k = 1-\tau(1-p) < 1 \tag{5.3}$$

Proof. When $n=1$ from the strict monotonicity of $\mathbf{B}_2\|\mathbf{u}_1 - \mathbf{u}_0\| \leqslant \tau^*(1-p)r$, then from (5.2) we obtain $W_1 \leqslant kW_0$, $W_n = \|Uu_n - f\|_{X^*}$, $n \geqslant 0$. By induction for $n \geqslant 2$: $\|\mathbf{u}_n - \mathbf{u}_{n-1}\|_X \leqslant \tau W_{n-1} C_4^{-1}$, $W_n \leqslant kW_{n-1}$, and then $\|\mathbf{u}_n - \mathbf{u}_0\|_X < r$ and inequalities (5.3) are true.

Suppose that the region of the body and the external loads are such that iterations (5.1) $\mathbf{u}_n \in X_1 \subset X$. From (5.1) for $\mathbf{u}^1, \mathbf{u}^2 \in X$, which differ only for $t \in [t_1, t_2]$:

$$|\Delta_1 b_{ij}(t)| \leqslant V\rho(\Delta V) \int_{t_1}^{t_2} |\Delta e_{ij}^{\cdot}(y)|\, dy, \quad \Delta \equiv t_2 - t_1, \quad \Delta e_{ij}^{\cdot} \equiv e_{ij}^{\cdot 1} - e_{ij}^{\cdot 2}, \quad V = \text{const} \tag{5.4}$$

Using the Hölder inequality, from (5.4) we have the estimate

$$\|\mathbf{B}(\mathbf{u}^1, \mathbf{u}^2) - \mathbf{Uu}^2\|_{X^*} \leqslant \mu(\Delta)\|\Delta u\|_X, \quad \Delta\mathbf{u} \equiv \mathbf{u}^1 - \mathbf{u}^2, \quad \mu(\Delta) \equiv \Delta V\rho(\Delta V)/3G \tag{5.5}$$

It follows from this that the operator $\mathbf{U}$ satisfies the conditions of L-continuity with constant $C_3^* = C_3 + \mu(\Delta)$ for such $\mathbf{u}^1$, $\mathbf{u}^2$. Moreover, from (5.5) and the strict monotonicity of $\mathbf{B}_2$ it follows that there exist Δ' and Δ''.

$$\mu(\Delta') = p \inf_{\substack{\mathbf{u}^1, \mathbf{u}^2 \in X_1 \\ \mathbf{u}^1 = \mathbf{u}^2 \text{ for } t \notin [t_1, t_2]}} \{\|\mathbf{B}(\mathbf{u}_1, \mathbf{u}^2) - \mathbf{Uu}^1\|_{X^*}/\|\Delta\mathbf{u}\|_X\}, \ \mu(\Delta'') = 2C_4, \ p<1$$

such that when $\Delta < \Delta'$ inequality (5.2) is satisfied, and when $\Delta < \Delta''$ the operator $\mathbf{U}$ is strictly monotonic with constant $C_4^* = C_4 - \mu(\Delta)/2 > 0$.

Then from Theorem 3.4 of [5, p. 104] and Lemma 5.1 we obtain the following:

Assertion 5.1. Iterations which are defined by the first (or second) rule of (5.1) with $\tau \in]0, 2C_4^*(C_3^*)^{-2}[$ (or $\tau \in]0, 1]$) converge to the exact solution $\mathbf{u}$ of (3.7), with the error estimate

$$\|\mathbf{u}_n - \mathbf{u}\|_X < \beta^n n^{\nu-1} C, \ C \equiv \tau_0 \|\mathbf{Uu}_0 - \mathbf{f}\|_{X^*} (1 + \tau_0 \mu(\Delta_0)/\beta)^{\nu-1}(1-\beta)^{-\nu}, \ \nu = T/\Delta_0$$
$$\beta \equiv [1 - 2C_4^*\tau + (C_3^*\tau)^2]^{1/2} < 1, \ \tau_0 = \tau, \ \Delta_0 = \Delta'' \ (\text{или } \beta = k < 1, \ \tau_0 = \tau/C_4, \ \Delta_0 = \Delta')$$

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DYNAMICS OF A HIGH-REVOLUTION COMPRESSOR†

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The dynamics of a high-revolution compressor where each of the mountings is formed by two single-row ball bearings pressed into a common housing and considered. Springs with a rated force are set up between the housing and the body. Relations are obtained between the mass characteristics of the housings, the coefficients of rigidity of the elastic mountings and the frequency of rotation of the compressor for which the dynamic pressures on the mountings of an unbalanced rotating compressor vanish. Formulas are obtained which define the first two critical frequencies of rotation of a compressor in elastic mountings.

As THE frequency of rotation increases, the operating life of ball bearings when they are rigidly installed in the framework falls sharply since the pressure between the balls of a bearing and its external ring increases in proportion to the square of the angular velocity of rotation. According to the theory which is presented in courses in theoretical mechanics [1–3], in order the reduce the pressure on the mountings, it is necessary to reduce the static and instantaneous imbalance of the rotating solid to zero. A whole branch of technology, that is balancing technology, has been set up for this purpose. However, in practice, as a consequence of deformation, the reaction of ball bearings, starting from a rather low value of the eccentricity and angle which characterizes the instantaneous imbalance, continues to increase sharply at high values of the frequency of rotation, which also leads to the destruction of the bearings in spite of very careful balancing [4].

The installation of elastic mountings [5] between the external ring of a bearing and its housing became an alternative when designing efficient high-revolution machines mounted on ball bearings. However, their premature breakdown is observed when the rotor is installed in single-row ball bearings due to the misalignment of the cage with respect to the external ring of the bearing. It is shown below that, when mountings consisting of two single row ball-bearings pressed into a common housing which is mounted elastically in the body are used, all the advantages of a shaft in elastic mountings are preserved and there is no skewing of the cage.

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